Problem 1377. Consider a decreasing and positive sequence $\{a_{2k-1}\}_{k\geq 1}$ whose sum converges. Prove there exists another decreasing and positive sequence $\{a_{2k}\}_{k\geq 1}$ such that the combined sequence $\{a_n\}_{n\geq 1}$ is decreasing and the series $\sum_{n=1}^{\infty} a_n$ converges to an irrational number

Proof. Define $a_{2n} = ca_{2n-1} + (1-c)a_{2n+1}$ for some $c \in [0,1]$. Let $S = \sum_{i=1}^{\infty} a_{2i-1}$ be the given sum. Note that $\sum_{i=1}^{\infty} a_{2i} = \sum_{i=1}^{\infty}(ca_{2n-1} + (1-c)a_{2n+1})$ is telescopic and convergent with some $S' = (c-1)a_1 + S$.

$$\begin{aligned} S + S' &= 2S + (c-1)a_1 \\ &= 2S - a_1 + c \end{aligned}$$

Notice that

$$2S - a_1 \leq 2S - a_1 + c \leq 2S \tag{1}$$

$$\exists w \in (2S - a_1, 2S) \cap (\mathbb{R} \setminus \mathbb{Q}) \tag{2}$$

Let $2S - a_1 + c = w$

$$c = 1 + \frac{w - 2S}{a_1} \tag{3}$$

$$a_1(c-1) = w - 2S \tag{4}$$

Since $c - 1 \in [-1, 0]$, $0 \geq w - 2S \geq -a_1 \implies 0 \leq c \leq 1$

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