Problem 882. Find all functions $f : \mathbb{R} \to \mathbb{R}$ where $f(x^4+y) = f(x)+f(y^4)$ for all $x, y \in \mathbb{R}$

Proof. We are going to show that the only function with the given property is the (constant) zero function.

Consider $x = y = 0$
$f(0+0) = f(0) + f(0)$
$f(0) = 2f(0)$
$f(0) = 0$

Now consider $y = 0$ and $x \in \mathbb{R}$
$f(x^4+0) = f(x) + f(0)$

$$f(x^4) = f(x), \forall x \in \mathbb{R} \tag{1}$$

Let $a \in \mathbb{R}$. Then $f(-a) = f((-a)^4) = f(a^4) = f(a)$, which means that the function is even.

Substituting equation 1 to the original equation, we get

$$f(x^4+y) = f(x^4) + f(y), \forall x, y \in \mathbb{R} \tag{2}$$

Let $z \in \mathbb{R}^+$. Then there exists $x \in \mathbb{R}$ such that $z = x^4$. Replacing x^4 with z in equation 2, we obtain

$$f(z+y) = f(z) + f(y), \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+ \tag{3}$$

If $b \in \mathbb{R} - \{0\}$, either b or $-b$ is positive. Substituting b and $-b$ in equation 3, we have $f(0) = f(-b) + f(b)$. Since $f(0) = 0$, $f(-b) = -f(b), \forall b \in \mathbb{R}$, which means the function is odd.
The only function that is both even and odd is $f = 0$.

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