Solution *by Steven Jang, Cal Poly Pomona Problem Solving Group, Pomona, CA*

Problem 889. If $h_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$, then identify:

$$\lim_{n\to\infty} (\log(2) - h_n)n \tag{i}$$

$$\lim_{n\to\infty} ((h_n h_{n+1} - \log^2(n))n) \tag{ii}$$

Proof. We are going to show that the limit (i) is equal to $\frac{1}{4}$.

We know $\log(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Thus,

$$\begin{aligned}
(\log(2) - h_n)n &= \frac{n}{2n+1} - \frac{n}{2n+2} + \frac{n}{2n+3} - ... \\
&= \frac{1}{2+\frac{1}{n}} - \frac{1}{2+\frac{2}{n}} + \frac{1}{2+\frac{3}{n}} - ... \\
&= \frac{\frac{1}{n}}{(2+\frac{1}{n})(2+\frac{2}{n})} - \frac{\frac{1}{n}}{(2+\frac{3}{n})(2+\frac{4}{n})} + ... \\
&= \frac{1}{n}\sum_{i=0}^{\infty} \frac{1}{(2+\frac{2i+1}{n})(2+\frac{2i+2}{n})}
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{\infty} \frac{1}{(2+\frac{2i+2}{n})^2} &\leq \frac{1}{n}\sum_{i=0}^{\infty} \frac{1}{(2+\frac{2i+1}{n})(2+\frac{2i+2}{n})} \\
&\leq \frac{1}{n}\sum_{i=0}^{\infty} \frac{1}{(2+\frac{2i}{n})^2} r \qquad (1) \\
&= \frac{1}{4n} + \frac{1}{n}\sum_{i=0}^{\infty} \frac{1}{(2+\frac{2i+2}{n})^2}
\end{aligned}$$

We will show that the left and right sides of the inequality have the same limit. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{(2+2x)^2}$ and let $t \in \mathbb{N}$. The right Riemann sum of f on interval $[0, t]$ divided into tn equal subintervals is given by

$$RRS(f, t, n) = \frac{1}{n}\sum_{i=0}^{tn-1} \frac{1}{(2+\frac{2i+2}{n})^2}$$

From basic calculus we know that

$$\int_0^{\infty} \frac{1}{(2+2x)^2}dx = \lim_{t\to\infty}\int_0^{t} \frac{1}{(2+2x)^2}dx = \frac{1}{4}$$

We have that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{\infty}\frac{1}{(2+\frac{2i+2}{n})^2} &= \lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{n}\sum_{i=0}^{tn-1}\frac{1}{(2+\frac{2i+2}{n})^2}\\
&= \lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{tn-1}\frac{1}{(2+\frac{2i+2}{n})^2}\\
&= \lim_{t\to\infty}\int_0^t \frac{1}{(2+2x)^2}dx\\
&= \frac{1}{4}.
\end{aligned}$$

To justify that we can change the limit order, note that the sequence $RRS(f,t,n)$ is increasing in n for every t, and also increasing in t for every n. Clearly, the right side of inequality (1) also has limit $\frac{1}{4}$. By the Squeeze Theorem, we can say that the integral is equal to all sums in the inequality (1). Therefore the answer is $\frac{1}{4}$.

■

Proof. Since $\lim_{n\to\infty} h_n = \frac{1}{4}$, it is clear that the requested limit (ii) is $-\infty$.
Assuming that parts (i) and (ii) are related, it is likely that the statement in part (i) contains a typo and should read

$$\lim_{n\to\infty}((h_n h_{n+1} - \log^2(2))n)$$

Note that

$$\begin{aligned}
\Big(h_n h_{n+1} - \log^2(2)\Big)n &= \Big(h_n h_{n+1} - h_n^2 + h_n^2 - \log^2(2)\Big)n\\
&= \Big(h_n(h_{n+1}-h_n) + (h_n - \log(2))(h_n + \log(2))\Big)n\\
&= nh_n(h_{n+1}-h_n) + n(h_n - \log(2))(h_n + \log(2)).
\end{aligned}$$

Clearly $\lim_{n\to\infty} nh_n(h_{n+1}-h_n) = 0$ and using the result in part (i) we get that

$$\lim_{n\to\infty}((h_n h_{n+1} - \log^2(n))n) = \frac{\log(2)}{2}$$

■